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Reference Sheet for Elementary Lattice Theory

Propositional Calculus

Metatheorem Any two theorems are equivalent; ' true ' is a theorem.
Equivales is the only equivalence relation that is associative $((p \equiv q) \equiv r) \;\equiv\; (p \equiv (q \equiv r))$, and it has identity true.

Discrepancy ' $\not\equiv$ ' is symmetric, associative, has identity ' false ', mutually associates with equivales $((p \not\equiv q) \equiv r) \;\equiv\; (p \not\equiv (q \equiv r))$, and mutually interchanges with it as well $(p \not\equiv q \equiv r) \;\equiv\; (p \equiv q \not\equiv r)$.

Implication has the alternative definition $p \Rightarrow q \;\equiv\; \neg p \vee q$, thus having true as both left identity and right zero; it distributes over $\equiv$ in the second argument, and is self-distributive; and has the properties

Shunting

$$p \wedge q \Rightarrow r \quad\equiv\quad p \Rightarrow (q \Rightarrow r)$$

Contrapositive

$$p \Rightarrow q \quad\equiv\quad \neg q \Rightarrow \neg p$$

Modus Ponens

$$p \wedge (p \Rightarrow q) \quad\equiv\quad p \wedge q$$
$$p \wedge (q \Rightarrow p) \quad\equiv\quad p$$
$$p \wedge (p \Rightarrow q) \quad\Rightarrow\quad q$$

It is an order relation generated by '$\mathsf{false} \Rightarrow \mathsf{true}$'; whence 'from false, follows anything': $\mathsf{false} \Rightarrow p$. Moreover it has the useful property "(3.62)": $p \Rightarrow (q \equiv r) \quad\equiv\quad p \wedge q \;\equiv\; p \wedge r$.

Conjunction and disjunction distribute over one another, $\vee$ distributes over $\equiv$, $\wedge$ distributes over $\equiv - \equiv$ in that $p \wedge (q \equiv r \equiv s) \quad\equiv\quad p \wedge q \;\equiv\; p \wedge r \;\equiv\; p \wedge s$, and they satisfy,

Excluded Middle
$p \vee \neg p$

Contradiction
$p \wedge \neg p \equiv \mathsf{false}$

Absorption
$p \wedge (\neg p \vee q) \equiv p \wedge q$
$p \vee (\neg p \wedge q) \equiv p \vee q$

De Morgan
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Lattices

The distributive lattice interface $(L, \sqsubseteq, \sqcap, \sqcup, \bot, \top)$ has the following implementations:

- Booleans: $(\mathbb{B}, \Rightarrow, \wedge, \vee, \mathsf{false}, \mathsf{true})$ —Our ambient logic!
- Extended Number Line: $(\mathbb{R}, \leq, \min, \max, -\infty, +\infty)$
- Naturals under division: $(\mathbb{N}, |, \mathsf{gcd}, \mathsf{lcm}, 1, 0)$
- Substructures of a given *datatype* with the substructure ordering.
  E.g., sets, lists, and graphs with subset, subsequence, and subgraph ordering.

An *order* is a relation $\sqsubseteq : L \to L \to \mathbb{B}$ satisfying the following three properties:

Reflexivity
$a \sqsubseteq a$

Transitivity
$a \sqsubseteq b \;\wedge\; b \sqsubseteq c \;\Rightarrow\; a \sqsubseteq c$

Antisymmetry
$a \sqsubseteq b \;\wedge\; b \sqsubseteq a \;\Rightarrow\; a = b$

An order is *bounded* if there are elements $\top, \bot : L$ being the lower and upper bounds of all other elements:

Top Element
$a \;\sqsubseteq\; \top$

Bottom Element
$\bot \;\sqsubseteq\; a$

A *lattice* is a pair of operations $\sqcap, \sqcup \;:\; L \to L \to L$ specified by the properties:

$\sqcup$-Characterisation
$a \sqsubseteq c \;\wedge\; b \sqsubseteq c \quad\equiv\quad a \sqcup b \sqsubseteq c$

$\sqcap$-Characterisation
$c \sqsubseteq a \;\wedge\; c \sqsubseteq b \quad\equiv\quad c \sqsubseteq a \sqcap b$

The operations act as providing the greatest lower bound, 'glb', 'supremum', or 'meet', by $\sqcap$; and the least upper bound, 'lub', 'infimum', or 'join', by $\sqcup$.
Let $\Box$ be one of $\sqcap$ or $\sqcup$, then:

Symmetry of $\Box$
$a \Box b \;=\; b \Box a$

Associativity of $\Box$
$(a \Box b) \Box c \;=\; a \Box (b \Box c)$

Idempotency of $\Box$
$a \Box a \;=\; a$

Zero of $\Box$
$a \sqcup \top = \top$
$a \sqcap \bot = \bot$

Identity of $\Box$
$a \sqcup \bot = a$
$a \sqcap \top = a$

Absorption
$a \sqcap (a \sqcup b) = a$
$a \sqcup (a \sqcap b) = a$

Self-Distributivity of $\Box$
$a \Box (b \Box c) \quad=\quad (a \Box b) \Box (a \Box c)$

Weakening / Strengthening
$a \;\sqsubseteq\; a \sqcup b$
$a \sqcap b \;\sqsubseteq\; a$
$a \sqcap b \;\sqsubseteq\; a \sqcup b$
$a \sqcup (b \sqcap c) \;\sqsubseteq\; a \sqcup b$
$a \sqcap b \;\sqsubseteq\; a \sqcap (b \sqcup c)$

Induced Defs. of Inclusion
$a \sqsubseteq b \quad\equiv\quad a \sqcup b \;=\; b$
$a \sqsubseteq b \quad\equiv\quad a \sqcap b \;=\; a$

Monotonicity of $\Box$
$a \sqsubseteq b \;\wedge\; c \sqsubseteq d \quad\Rightarrow\quad a \Box c \;\sqsubseteq\; b \Box d$

Golden Rule
$a \sqcap b \;=\; a \quad\equiv\; b = a \sqcup b$
$a \sqcap b \;=\; a \sqcup b \;\equiv\; a = b$
$a \sqcup b \;\sqsubseteq\; a \sqcap b \;\equiv\; a = b$

Duality Principle:
If a statement S is a theorem, then so is $S[(\sqsubseteq, \sqcap, \sqcup, \top, \bot) := (\sqsupseteq, \sqcup, \sqcap, \bot, \top)]$.